Simulation of trawl processes using SINN architectures

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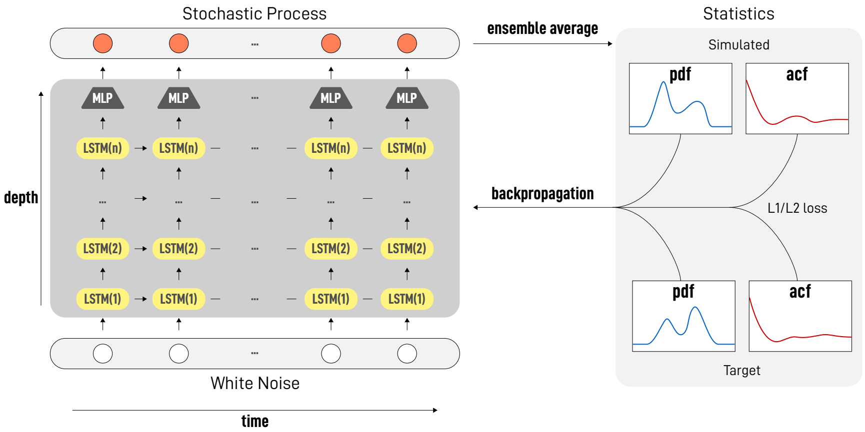
Ambit processes and their simulation

Ambit processes are a class of stochastic processes that generalize many known models and are suitable for modeling complex spatio-temporal phenomena.

- This class includes Levy processes, fractional Brownian motion, Lévy-driven Volterra processes etc
- Application areas: turbulence physics, meteorology and climate, financial mathematics
- Known simulation methods require a lot of expensive computations related to inversion of the Fourier integrals

SINN architectures



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Puprose of the work

The aim of the work is to investigate the applicability of SINN models for modeling ambit-processes

- The disadvantage of the original approach is that another simulation method is required for training.
- Most ambit-processes are non-stationary, and their distributions do not have closed forms in the form of densities/probability functions; however, their Fourier transforms can be expressed explicitly
- We limit ourselves to a special case — trawl processes, and try to construct a suitable loss function based on characteristic functions

Trawl processes

- Informally speaking, trawl process is a random measure of a moving geometric figure in a higher-dimensional space. Trawl processes always stationary and ergodic
- More formally, consider a random measure L (of particular kind) and a set $A \subset \mathbb{R} \times (-\infty; 0)$; The trawl process X_t is defined as $L(A_t)$, where $A_t = A + (0, t)$
- Special cases include Ornstein-Uhlenbeck process, fractional Ornstein-Uhlenbeck process and in general any stationary Gaussian process with differentiable autocovariance function that is non-increasing and convex

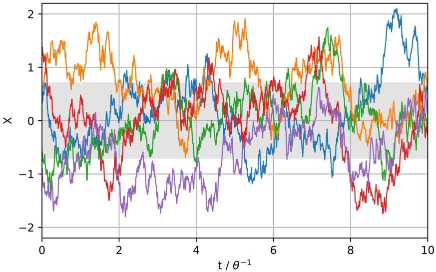
Special case: OU-process

Ornstein–Uhlenbeck process is defined to be the stationary solution of the SDE

$$dX_t = \lambda(\mu - X_t)dt + \sigma dW_t$$

It can be represented as trawl process with L being standard on $\mathbb{R}^2$ white noise $L(dx, dt) = W(dx, dt)$ and trawl set given by

$$A = \left\{(x, t) \;\middle|\; 0 \le x \le \frac{\sigma^2}{2} \cdot \exp(\lambda t),\;\; t < 0\right\}$$



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Naive variant of CF-based loss

Consider now the problem of the learning parameters θ of a SINN model to simulate the distribution $\boldsymbol{X}_{\boldsymbol{t}}$ of the process X_t at times $\boldsymbol{t} = (t_1, \ldots, t_d)$; Define

- $\varphi_{X_t}(\boldsymbol{u}, \boldsymbol{t})$ — analytical / MC-estimated CF of the $\boldsymbol{X}_{\boldsymbol{t}}$
- $\varphi_{\mathrm{sinn}(\theta)}(\boldsymbol{u}, \boldsymbol{t})$ — MC-estimate of the CF of the distribution originated by SINN model

Suggested loss approximates $\int_{\mathbb{R}^d} \left|\varphi_{X_t}(\boldsymbol{u}, \boldsymbol{t}) - \varphi_{\mathrm{sinn}(\theta)}(\boldsymbol{u}, \boldsymbol{t})\right| d\boldsymbol{u}$:

$$\mathcal{L}_{\mathrm{cf}}(\theta, \boldsymbol{t}) = \int_{[-T;T]^d} \left|\varphi_{X_t}(\vec{u}) - \varphi_{\mathrm{sinn}(\theta)}(\vec{u})\right| du \approx$$

$$\approx \frac{2^d T^d}{N} \sum_{i=1}^{N} \left|\varphi_{X_t}(\vec{u}_i) - \varphi_{\mathrm{sinn}(\theta)}(\vec{u}_i)\right|, \quad \vec{u}_i \overset{\text{i.i.d}}{\sim} U([-T;T]^d)$$

General schema for CF-based loss

Consider set of multi-indexes $\mathcal{T} \subset [d] = \{1, \dots, d\}$; General loss is given by:

$$\mathcal{L}(\theta) = \sum_{\boldsymbol{i} \in \mathcal{T}} \alpha_{\boldsymbol{i}} \cdot \mathcal{L}_{\mathrm{cf}}(\theta, \boldsymbol{t}_{\boldsymbol{i}})$$

Notable special cases:

- $\mathcal{T} = \{(1, \dots, d)\}$ — joint-distribution
- $\mathcal{T} = \{(1), \dots, (d)\}$ — all marginal distributions
- $\mathcal{T} = \{(1, 2), (1, 3) \dots, (d - 1, d)\}$ — all possible pairwise distributions

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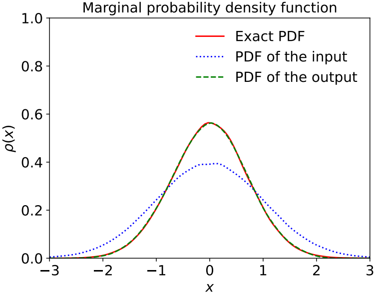
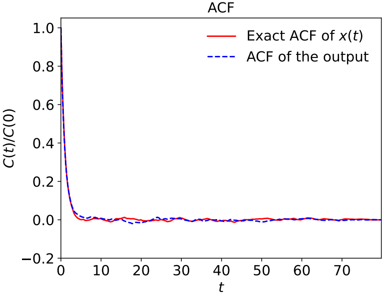
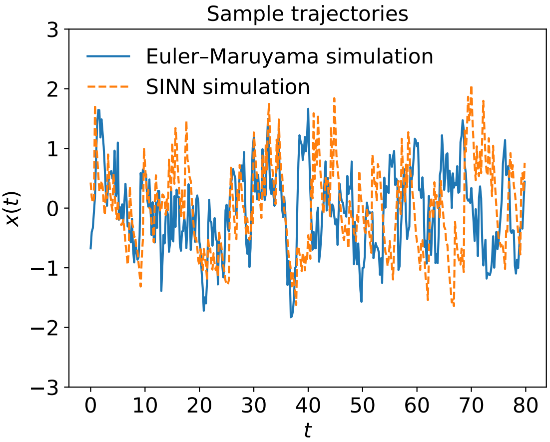
Simulation-based approach for OU-process

- SINN depth: 1, time grid size: 400, discretization step: 0.1
- Scheme: full loss with CF empirically estimated from Euler–Maruyama simulations
- T4 x 2 GPU, learning time $\approx$ 15 min



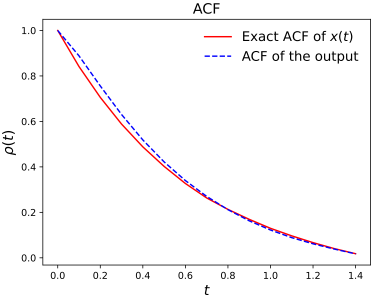


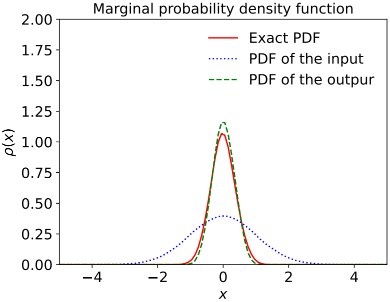
Simulation-based approach for OU-process



Simulation-free approach for OU-process with CF-based loss

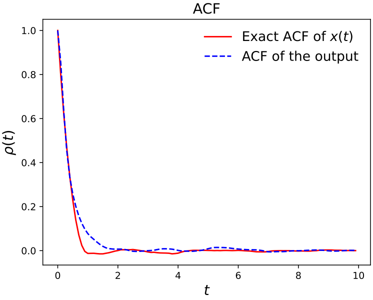
- SINN depth: 2, time grid size: 15, discretization step: 0.1
- Scheme: pairwise + marginal loss
- T4 x 2 GPU, learning time ≈ 15 min

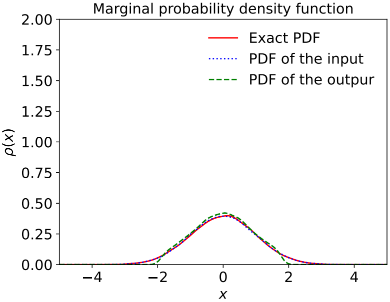




CF-based loss for short-memory trawl process

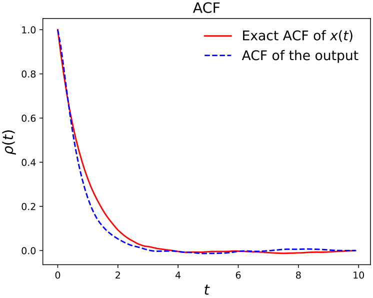
- SINN depth: 2, time grid size: 100, discretization step: 0.1
- Scheme: rolling window + marginal loss + selected pairs
- T4 x 2 GPU, learning time ≈ 25 min

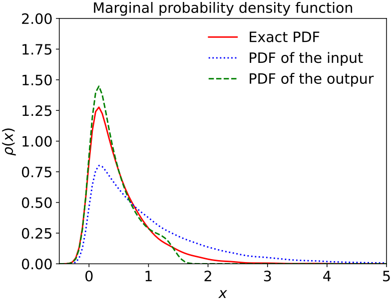




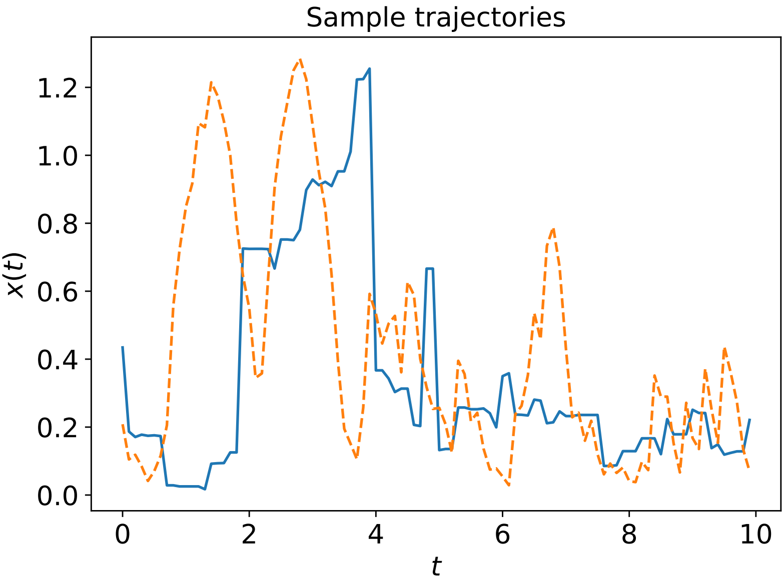
CF-based loss for gamma trawl process

- SINN depth: 2, time grid size: 100, discretization step: 0.1; exponential distribution as input
- Scheme: rolling window + marginal loss + selected pairs
- T4 x 2 GPU, learning time ≈ 25 min





CF-based loss for gamma trawl process: trajectories



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Conclusion

- Suggested simulation-free approach based on characteristic functions allows for simulation for trawl process; However heuristics for choosing of index schema for loss function should be developed
- It seems that current architecture does not suit for simulation of processes with short memory — SINN cannot learn zero autocorrelation
- Further research includes generalization of this approach to another classes of the ambit processes (BSS/LSS process, VMLV-process, weighted trawl process)

Ambit-fields and ambit-processes

Ambit-field is a random field of the form

$$Y_t(\boldsymbol{x}) = \int\limits_{A_t(\boldsymbol{x})} h(\boldsymbol{x}, t; \boldsymbol{y}; s)\sigma_s(\boldsymbol{y})L(d\boldsymbol{y}; ds)$$

and ambit process is a process of the form $Y_t(\gamma(t))$ where $\gamma(t)$ is some curve in $\mathbb{R}^d$; In the formula above

- $L(d\boldsymbol{y}, ds)$ — random measure defined on space-time $\mathbb{R}^d \times \mathbb{R}$
- $\sigma_s(\boldsymbol{y})$ — stochastic volatility/intermittency field
- $h(\boldsymbol{x}, t; \boldsymbol{y}; s)$ — determensitic function (kernel)
- $A_t(\boldsymbol{x})$ – ambit set of the point $(\boldsymbol{x}, t)$ in the space-time, i.e. set of all points which can influence on the value of $Y_t(\boldsymbol{x})$

Trawl processes

Trawl process defined as ambit process of particular kind:

$$X_t = \int_{\mathbb{R}\times(-\infty:t)} \mathbb{1}_A(\boldsymbol{y}, t-s)L(d\boldsymbol{y}, ds)$$

where

- L is stationary Levy basis on $\mathbb{R}^1 \times \mathbb{R}$ (meaning that distribution of $L(A)$ is translation invariant)
- $A \subset \mathbb{R} \times (-\infty; 0)$ is some set called *trawl set*

Informally speaking, trawl process is a random measure of a moving geometric figure in a higher-dimensional space

Characteristic functions

Consider some process X_t and vector of time moments $\boldsymbol{t} = (t_1, \dots, t_d)$;
For the distribution of the $\boldsymbol{X_t} = (X_{t_1}, \dots, X_{t_d})$ it's **characteristic functions** defined as

$$\varphi_{X_t}(\boldsymbol{u}, \boldsymbol{t}) = \mathbb{E}\left[e^{i\langle \boldsymbol{X_t}, \boldsymbol{u}\rangle}\right]$$

- Characteristic function uniquely determines distribution
- Every stochastic process is uniquely characterized by it's finite-dimensional distributions

Rainfall exceedance according to Met Office MIDAS

Model: Λ_t is gamma-trawl process with seed $L' \sim \Gamma(\alpha, \beta)$ and autocorrelation function $\rho(h) = \exp(-rh)$. The model is defined as zero-inflated mixture of exponential distributions:

$$X_j \sim \begin{cases} 0 & \text{with probability } 1 - \exp(-\kappa\Lambda_j) \\ \mathrm{Exp}(\kappa\Lambda_j) & \text{with probability } \exp(-\kappa\Lambda_j) \end{cases}$$

Data: